SEACAS Theory Manuals: Part I.
Problem Formulation in Nonlinear Solid Mechanics

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Abstract

This report gives an introduction to the basic concepts and principles involved in the formulation of nonlinear problems in solid mechanics. By way of motivation, the discussion begins with a survey of some of the important sources of nonlinearity in solid mechanics appliations, using wherever possible simple one dimensional idealizations to demonstrate the physical concepts. This discussion is then generalized by presenting generic statements of initial/boundary value problems in solid mechanics, using linear elasticity as a template and encompassing such ideas as strong and weak forms of boundary value problems, boundary and initial conditions, and dynamic and quasistatic idealizations. The notational framework used for the linearized problem is then extended to account for finite deformation of possibly inelastic solids, providing the context for the descriptions of nonlinear continuum mechanics, constitutive modeling, and finite element technology given in three companion reports.

Formulation of Nonlinear Problems:
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Introduction

Introduction

We begin our study of nonlinear computational solid mechanics in this chapter by surveying some frequently encountered sources of nonlinearity in engineering mechanics. This will be done in a rather elementary way by discussing the truss member, which is perhaps the simplest structural idealization and which is assumed to transmit loads in the axial direction only. By introducing various nonlinearities into this system one at a time, we will motivate the more general discussion of nonlinear continuum mechanics, constitutive modeling, and numerical treatments to follow. This model system will serve as a template throughout the text as new continuum mechanical and computational ideas are introduced.

Following this motivation will be an introduction to the prescription of initial/boundary value problems in solid mechanics. This introduction will be provided by discussing a completely linear system; namely, linear elastic behavior in a continuum subject to infinitesimal displacements. This treatment will include presentation of the relevant field equations, boundary conditions, and initial conditions, encompassing both dynamic and quasistatic problems in the discussion. Also featured is a brief discussion of the "weak" or "integral" form of the governing equations, providing a starting point for application of the finite element method. Examination of these aspects of problem formulation in the comparatively simple setting of linear elasticity allows one to concentrate on the ideas and concepts involved in problem description without the need for an overly burdensome notational structure.

In anticipation of nonlinear solid mechanics applications, however, we will find it necessary to expand this notational framework so that large deformation of solids can be accommodated. Fortunately, provided certain interpretations are kept in mind, the form of the governing equations is largely unchanged by the generalization of the linear elastic system. This chapter therefore provides an introduction to how this generalization can be made. However, it will be seen that the continuum description and constitutive modeling of solids undergoing large deformations are complex topics that should be understood in detail before accompanying numerical strategies are formulated and implemented. The closely related topics of nonlinear continuum mechanics and constitutive modeling will therefore be the subjects of the following two chapters, with significant discussion of numerical strategies being deferred to Finite Elements.

This introduction is concluded with a short list of references the reader may find useful as background material. Throughout the text we assume little or no familiarity with either the finite element method or nonlinear solid mechanics, but we do assume a basic level of familiarity with the mechanics of materials, linear continuum mechanics, and linear elasticity. Accordingly, these basic references are intended for those readers wishing to fill gaps in knowledge.

Other Reading

The following resources are suggested for those readers wishing to reinforce their knowledge of linear elasticity, elementary continuum mechanics, and/or fundamentals of solid mechanics. They are presented in alphabetical order, with no other significance to be attached to the order of presentation.

Fung, Y.C., 1965

Fung, Y.C., 1977

Hughes, T.J.R., 1987

Lai, W.M, Rubin, D. and Krempl, E., 1993

Malvern, L.E., 1969

Pilkey, W.D. and Wunderlich, W., 1994

Timoshenko, S. and Goodier, J.N., 1970

Nonlinear Behavior

Linear Structural Component

We consider the simple axial (or in structural terms, truss) member shown schematically in Figure 1.1. We can think of this member as a straight bar of material whose transverse dimensions are small compared to its overall length and which can only transmit loads in the axial direction. Real world examples include taut cables in tension, truss members, and similar rod-like objects.
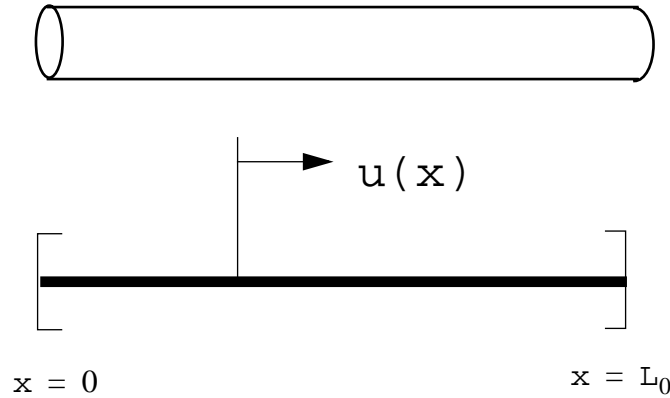
Figure 1.1   Axial model problem: schematic and local coordinate system.

We index the material with coordinates x, which run between values 0 and L_0. Assuming that all displacement of the rod occurs in the axial direction, we write this displacement as $u(x,t)$, with t signifying time. The *infinitesimal*, or *engineering*, *strain* at any point $x \in (0, L_0)$ is given by

$$\varepsilon_E(x, t) = \frac{\partial}{\partial x} u(x, t).$$

(1.1)

The *true stress* σ_T at any point in the bar and at any instant is described via

$$\sigma_T(x, t) = \frac{P(x, t)}{A(x, t)},$$

(1.2)

where P is the total axial force acting at location x, and A is the current cross-sectional area at that location. If the cross-sectional area does not change very much as a result of the deformation, it is appropriate to define the *nominal*, or *engineering*, *stress* as

$$\sigma_E = \frac{P(x, t)}{A_0(x)},$$

(1.3)

where $A_0(x)$ is the initial cross-sectional area at point x. If the material behaves in a *linear elastic* manner, then σ_E and ε_E are related via

$$\sigma_E = E\varepsilon_E, \tag{1.4}$$

where E is the *elastic modulus*, or *Young's modulus*, for the material.

To begin we consider the case of *static equilibrium* where inertial effects are either negligible or nonexistent, and the response is, therefore, independent of time. One can in this case suppress the time argument in Eqs. (1.2) and (1.4). The balance of linear momentum for the static case is expressed at each point x by

$$\frac{d}{dx}(A_0(x)\sigma_E(x)) = f(x), \tag{1.5}$$

where f is the *applied external loading*, assumed to be axial, with units of force per unit length. Substitution of Eq. (1.4) into (1.5) gives the following ordinary differential equation for $u(x)$ on the domain $(0, L_0)$:

$$\frac{d}{dx}\left(EA_0\frac{d}{dx}(u)\right) = f. \tag{1.6}$$

If we assume that the cross-section is uniform, so that A_0 does not vary with x, and that the material is *homogeneous*, so that E does not vary throughout the rod, one gets further simplification:

$$EA_0\frac{d^2}{dx^2}u(x) = f. \tag{1.7}$$

We note that (1.7) is a linear, second order differential equation for the unknown displacement field u. To pose a mathematical problem that can be uniquely solved, it is necessary to pose two *boundary conditions* on the unknown u. We will be interested primarily in two types, corresponding to *prescribed displacement* and *prescribed force* (or stress) boundary conditions. An example of the former would be

$$u(0) = \bar{u}, \tag{1.8}$$

while an example of the latter is

$$\sigma_E(L_0) = E\frac{du}{dx}(L_0) = \bar{\sigma}, \tag{1.9}$$

where $\bar{u}$ and $\bar{\sigma}$ are prescribed values for the displacement and axial stress at the left and right bar ends, respectively. In mathematics parlance the type of boundary condition in (1.8) is called a *Dirichlet* boundary condition, while the sort of boundary condition represented by (1.9) is a

Neumann boundary condition. Dirichlet boundary conditions involve the unknown dependent variable itself, while Neumann boundary conditions are expressed in terms of its derivatives.

Virtually any combination of such boundary conditions can be applied to our problem, but only one boundary condition (i.e., either a Neumann or Dirichlet condition) can be applied at each endpoint. In the case where Neumann (stress) conditions are applied at both ends of the bar, the solution $u(x)$ is only determinable up to an arbitrary constant (the reader may wish to verify this fact by applying separation of variables to Eq. (1.7)).

We now consider a particular case of this linear problem we have posed that will be useful in considering some of the various nonlinearities to be discussed below. In particular, suppose $f = 0$ on the domain $(0, L_0)$, and furthermore consider the boundary conditions

$$u = 0 \text{ at } x = 0 \tag{1.10}$$

and

$$\sigma_E = \frac{F^{ext}}{A_0} \text{ at } x = L, \tag{1.11}$$

where F^{ext} is an applied force on the right end of the rod.

In this case examination of Eq. ,,,(1.5) yields

$$A_0 \frac{d}{dx}(\sigma_E(x)) = 0, \tag{1.12}$$

meaning that σ_E does not vary along the length of the rod. Since σ_E is proportional to ε_E (see Eq. (1.4)), the strain must also be a constant value along the rod length. Finally, in view of Eq. (1.1), we conclude that $u(x)$ must vary linearly with x. In other words, we know that the solution $u(x)$ must take the form

$$u(x) = u(0) + \delta x = \delta x, \tag{1.13}$$

where δ is the elongation or difference between the left and right end displacement. The problem therefore reduces to finding the elongation produced by the applied force F^{ext}.

This problem is trivially solved and leads to the familiar linear relationship between F^{ext} and δ:

$$\frac{EA_0}{L_0}\delta = F^{ext}; \tag{1.14}$$

in other words, we have a simple linear spring with stiffness $\frac{EA_0}{L_0}$. After solving for δ one may merely substitute into (1.13) to obtain the desired expression for $u(x)$.

Material Nonlinearity

We can examine the case of a so-called *material nonlinearity* by replacing Eq. (1.4) with the following generic relationship between σ_E and ε_E:

$$\sigma_E = \hat{\sigma}(\varepsilon_E), \tag{1.15}$$

where $\hat{\sigma}$ is a smooth and generally nonlinear function (see Figure 1.2).

We make few restrictions on the specific form of $\hat{\sigma}$ other than to assume that $\dfrac{d}{d\varepsilon}\hat{\sigma} > 0$ for all values of ε_E. If we retain the assumption that $f = 0$ and impose boundary conditions (1.10) and (1.11), then Eq. (1.12) is still valid, that is:

$$\sigma_E = \frac{F^{ext}}{A_0} \tag{1.16}$$

throughout the rod.

Furthermore, since we assume that a one-to-one relation exists between σ_E and ε_E, we can conclude that just as in the linear case, the strain is a constant value in the rod given by

$$\varepsilon_E = \frac{\delta}{L_0}. \tag{1.17}$$
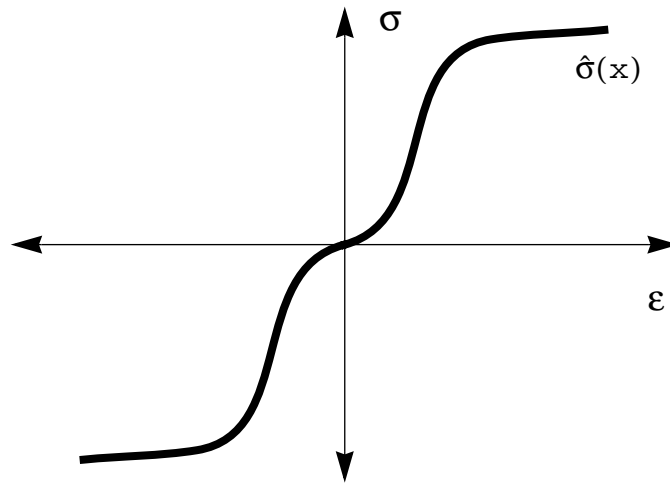
Figure 1.2   Schematic of a nonlinear, one-dimensional stress-strain relation.

We can solve the problem by finding δ as before, but now we must solve the nonlinear equation

$$A_0\hat{\sigma}\left(\frac{\delta}{L_0}\right) = F^{ext}. \tag{1.18}$$

Let us reexpress Eq. (1.18) as an equation for the displacement at the right end, which we shall denote as $d_L = u(L)$. We can write

$$N(d_L) = F^{ext},$$

(1.19)

where $N(d_L)$ is a nonlinear function of the unknown d_L defined in this case as

$$N(d_L): = A_0 \hat{\sigma}\left(\frac{d_L}{L_0}\right).$$

(1.20)

In general, Eq. (1.20) will not have a closed-form solution, and some sort of iterative procedure is necessary. Among the most common and widely used of such procedures is *Newton-Raphson iteration*. In this method one introduces a set of indices, i, corresponding to the iterations and given a current iterate, d_L^i, a first-order Taylor series expansion of (1.20) is utilized to generate the next iterate, d_L^{i+1}:

$$0 = F^{ext} - N(d_L^{i+1}) \approx F^{ext} - \left(N(d_L^i) + \frac{d}{dd_L}N(d_L^i)\Delta d_L\right),$$

(1.21)

where

$$d_L^{i+1} = d_L^i + \Delta d_L.$$

(1.22)

Equation (1.21) can be expressed more compactly via

$$K(d_L^i)\Delta d_L = R(d_L^i),$$

(1.23)

where $R(d_L^i)$, the *residual* or *out-of-balance force*, is given by

$$R(d_L^i): = F^{ext} - N(d_L^i),$$

(1.24)

and $K(d_L^i)$, the *incremental* or *tangent stiffness*, is written as

$$K(d_L^i): = \frac{d}{dd_L}N(d_L^i).$$

(1.25)

The Newton-Raphson procedure is then carried out by recursively solving Eqs. (1.23) and (1.22). We will return to this general algorithmic strategy, and variants of it, repeatedly throughout the text.

Geometric Nonlinearity

Geometric nonlinearities are induced by nonlinearities in the kinematic description of the system at hand. We will identify and work with several nonlinearities of this general type throughout the text, but to begin we will consider two particular cases, still working in the context of our simple model problem.

The first type of nonlinearity we consider is introduced by the use of nonlinear strain and stress measures in definition of the stress-strain relation. As an example let us consider alternatives to Eqs. (1.1) and (1.3), which defined the engineering strain ε_E and engineering stress σ_E that we have utilized to this point. When used in our model problem with $f = 0$ and boundary conditions (1.10) and (1.11), we have seen that the engineering strain does not vary over the rod's length, having the constant value $\frac{\delta}{L_0}$. The appropriateness of this strain measure depends upon the amount of deformation; specifically δ should be infinitesimal for this measure to be appropriate. In the presence of larger deformations, the *true*, or *logarithmic*, strain is often used:

$$\varepsilon_T = \int_{L_0}^{L} \frac{d\gamma}{\gamma} = \log\left(\frac{L}{L_0}\right) = \log(1 + \varepsilon_E). \tag{1.26}$$

Similarly if the cross-sectional area A changes appreciably during the process, it is likely that the engineering stress σ_E should be replaced by the true stress σ_T defined in Eq. (1.2). In the case of our model problem, this would imply

$$\sigma_T = \frac{F^{ext}}{A}, \tag{1.27}$$

where A is to be interpreted as the cross-sectional area in the final (deformed) configuration.

Relating this area to the elongation δ requires a constitutive assumption to be made. For example, if we assume the rod consists of an isotropic elastic material, we could approximate this variation by considering the area to vary according to Poisson's effect. This would require that for each differential increment $d\varepsilon_T$ in the axial true strain, each lateral dimension should be changed by a factor of $(1 - \nu d\varepsilon_T)$, where ν is *Poisson's ratio* for the material. At a given instant of the loading process, therefore, an incremental change in the area A can be approximated via

$$A + dA = (1 - \nu d\varepsilon_T)^2 A \approx (1 - 2\nu d\varepsilon_T)A. \tag{1.28}$$

Integrating (1.28) between the initial area A_0 and the final area and using (1.26) gives

$$A = A_0\left(\frac{L_0}{L}\right)^{2\nu} = A_0\left(\frac{L_0}{L_0 + \delta}\right)^{2\nu}. \tag{1.29}$$

If we assume that

$$\sigma_T = E\varepsilon_T,\tag{1.30}$$

then we can use Eqs. (1.26), (1.27), and (1.29) to conclude that

$$EA_0\left(\frac{L_0}{L_0 + \delta}\right)^{2\nu}\log\left(\frac{L_0 + \delta}{L_0}\right) = F^{ext},\tag{1.31}$$

which is an obviously nonlinear equation governing the elongation δ. Note that this nonlinearity is not caused by any sort of nonlinear stress-strain relation but instead results from the observation that the amount of deformation may not be small, necessitating more general representations of stress and strain.

The second sort of nonlinearity we wish to consider is that caused by large superimposed rigid body rotations and translations that introduce nonlinearities into many problems even when the strains introduced into the material are well-approximated by infinitesimal measures. Toward this end we refer to Figure 1.3, in which we imbed our one-dimensional truss element in a two-dimensional frame. We locate one end of the rod at the origin and consider this end to be pinned so that it is free to rotate but not to translate. The other rod end, initially located at coordinates (x_1^0, x_2^0), is subjected to a (vector valued) force $\mathbf{F}^{ext}$, which need not be directed along the axis of the rod.
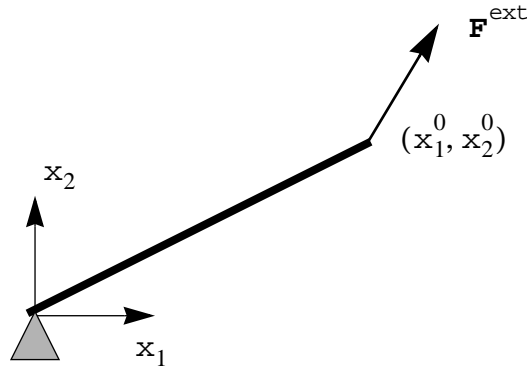
Figure 1.3   Model problem with infinitesimal motions superposed on large rigid body motions.

We note that if we placed a restriction of small motions this problem would be ill-posed; since the rod is incapable of transmitting anything but axial force, $\mathbf{F}^{ext}$ would need to act in the axial direction in this case. In the current context we allow unlimited rotation to take place with the result being, of course, that the rod will rotate until it aligns with $\mathbf{F}^{ext}$ in its equilibrium condition. In fact, this observation allows us to guess the solution to the problem. Since we assume that the axial response of the rod is completely linear, we may deduce that the final elongation is given by

$$\delta = \frac{L_0 \|\mathbf{F}^{\mathrm{ext}}\|}{E A_0},$$ (1.32)

where $\|\mathbf{F}^{\mathrm{ext}}\|$ denotes the Euclidean length of the vector $\mathbf{F}^{\mathrm{ext}}$. The final orientation of the rod must coincide with the direction of $\mathbf{F}^{\mathrm{ext}}$, so we can write the final position of the rod end using the coordinates (x_1^f, x_2^f) as:

$$\begin{bmatrix} x_1^f \\ x_2^f \end{bmatrix} = \frac{L_0}{\|\mathbf{F}^{\mathrm{ext}}\|} \left(1 + \frac{\|\mathbf{F}^{\mathrm{ext}}\|}{E A_0} \right) \begin{bmatrix} F_1^{\mathrm{ext}} \\ F_2^{\mathrm{ext}} \end{bmatrix};$$ (1.33)

or writing the solution in terms of the rod end displacements d_1 and d_2:

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{L_0}{\|\mathbf{F}^{\mathrm{ext}}\|} \left(1 + \frac{\|\mathbf{F}^{\mathrm{ext}}\|}{E A_0} \right) \begin{bmatrix} F_1^{\mathrm{ext}} \\ F_2^{\mathrm{ext}} \end{bmatrix} - \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}.$$ (1.34)

It is instructive to proceed as though we do not know the solution summarized in (1.34) and formulate the equilibrium equations governing d_1 and d_2.

If we observe that the elongation δ of the rod can be written as

$$\delta = \sqrt{(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2} - L_0,$$ (1.35)

then Eq. (1.32) gives the relationship between $\|\mathbf{F}^{\mathrm{ext}}\|$ and the unknown displacements. Furthermore, as noted above, the direction of $\mathbf{F}^{\mathrm{ext}}$ is given by

$$\frac{\mathbf{F}^{\mathrm{ext}}}{\|\mathbf{F}^{\mathrm{ext}}\|} = \frac{1}{\sqrt{(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2}} \begin{bmatrix} d_1 + x_1^0 \\ d_2 + x_2^0 \end{bmatrix}.$$ (1.36)

Combining these facts gives the equation that governs d_1 and d_2:

$$\begin{bmatrix} F_1^{\mathrm{ext}} \\ F_2^{\mathrm{ext}} \end{bmatrix} = E A_0 \frac{\sqrt{(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2} - L_0}{L_0 \sqrt{(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2}} \begin{bmatrix} d_1 + x_1^0 \\ d_2 + x_2^0 \end{bmatrix}.$$ (1.37)

The reader may wish to verify this equation by substituting the solution (1.34) into (1.37).

Equation (1.37) is a nonlinear, vector-valued equation for the unknowns d_1 and d_2. Recalling the generic form for nonlinear equations we introduced in the one-dimensional case in (1.19), we could write this generically as

$$\mathbf{N}(\mathbf{d}) = \mathbf{F}^{\text{ext}}, \tag{1.38}$$

where

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{1.39}$$

and

$$\mathbf{N}(\mathbf{d}) = EA_0 \frac{\sqrt{\left(d_1 + x_1^0\right)^2 + \left(d_2 + x_2^0\right)^2} - L_0}{L_0 \sqrt{\left(d_1 + x_1^0\right)^2 + \left(d_2 + x_2^0\right)^2}} \begin{bmatrix} d_1 + x_1^0 \\ d_2 + x_2^0 \end{bmatrix}. \tag{1.40}$$

Just as was done in the last section for the one degree of freedom case, we could introduce a Newton-Raphson strategy to solve (1.38) via

$$\mathbf{K}(\mathbf{d}^i)\Delta\mathbf{d} = \mathbf{R}(\mathbf{d}^i) = \mathbf{F}^{\text{ext}} - \mathbf{N}(\mathbf{d}^i) \tag{1.41}$$

and

$$\mathbf{d}^{i+1} = \mathbf{d}^i + \Delta\mathbf{d}, \tag{1.42}$$

where

$$\mathbf{K}(\mathbf{d}^i) = \frac{\partial \mathbf{N}}{\partial \mathbf{d}}(\mathbf{d}^i) = \begin{bmatrix} \dfrac{\partial N_1}{\partial d_1} & \dfrac{\partial N_1}{\partial d_2} \\ \dfrac{\partial N_2}{\partial d_1} & \dfrac{\partial N_2}{\partial d_2} \end{bmatrix}\Bigg|_{\mathbf{d} = \mathbf{d}^i}. \tag{1.43}$$

Carrying out the calculation of $\mathbf{K}(\mathbf{d}^i)$ for the specific $\mathbf{N}(\mathbf{d})$ at hand gives

$$\mathbf{K}(\mathbf{d}^i) = \mathbf{K}_{\text{direct}}(\mathbf{d}^i) + \mathbf{K}_{\text{geom}}(\mathbf{d}^i). \tag{1.44}$$

$\mathbf{K}_{\text{direct}}(\mathbf{d}^i)$ is given by

$$\mathbf{K}_{\text{direct}}(\mathbf{d}^i) = \cfrac{\cfrac{A_0 E}{[(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2]^{\frac{3}{2}}} \times}{\begin{bmatrix} (d^i{}_1 + x_1^0)^2 & (d^i{}_1 + x_1^0)(d^i{}_2 + x_2^0) \\ (d^i{}_1 + x_1^0)(d^i{}_2 + x_2^0) & (d^i{}_2 + x_2^0)^2 \end{bmatrix}} ; \tag{1.45}$$

and $\mathbf{K}_{\text{geom}}(\mathbf{d}^i)$ is

$$\mathbf{K}_{\text{geom}}(\mathbf{d}^i) = A_0 E \left(\frac{1}{L_0} - \frac{1}{\sqrt{(d_1 + x_1^0)^2 + (d_2 + x_2^0)^2}} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{1.46}$$

As the notation suggests, $\mathbf{K}_{\text{direct}}$ is sometimes referred to as the *direct stiffness* or that part of the stiffness emanating directly from the material stiffness of the system at hand. $\mathbf{K}_{\text{geom}}(\mathbf{d}^i)$, on the other hand, is sometimes called the *geometric stiffness* and arises not from the inherent stiffness of the material but by virtue of the large motions that the current problem allows.

To gain some insight into these issues in the current context, consider the case where $\left\| \mathbf{d}^i \right\| \ll \left\| \mathbf{x}^0 \right\|$; i.e., the case where the motions are small in comparison to the rod's length. In this case we find

$$\mathbf{K}_{\text{geom}}(\mathbf{d}^i) \to 0 , \tag{1.47}$$

and

$$\mathbf{K}_{\text{direct}}(\mathbf{d}^i) \to \frac{A_0 E}{L_0} \begin{bmatrix} \cos\Theta \cos\Theta & \cos\Theta \sin\Theta \\ \cos\Theta \sin\Theta & \cos\Theta \cos\Theta \end{bmatrix} , \tag{1.48}$$

where $\Theta = \text{atan}\left(\dfrac{x_2^0}{x_1^0} \right)$ is the angle between the original axis of the truss member and the positive x-axis. In other words, when the motions become small, the geometric stiffness vanishes, and the direct stiffness becomes the familiar stiffness matrix associated with a two-dimensional truss member.

Contact Nonlinearity

A final type of nonlinearity we wish to consider is that created due to contact with another deformable or rigid entity. As a simple model problem for this case, we refer to Figure 1.4, where we consider a prescribed motion $\bar{d}$ of the left end of our one-dimensional rod and consider the unknown displacement d of the right end to be subject to the constraint

$$g(d) = d - g_0 \leq 0, \tag{1.49}$$

where g_0 is the initial separation, or *gap*, between the right rod end and the rigid obstacle.
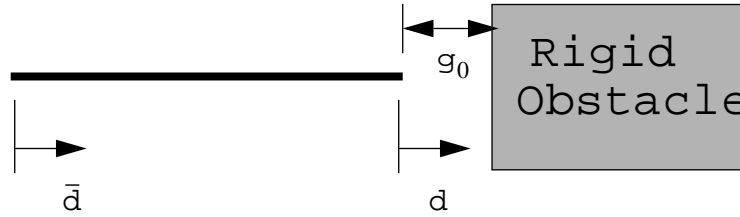
Figure 1.4   Schematic of the rigid obstacle problem.

Even if we assume that the motions are small and the material response of the rod is elastic, the equations governing the response of our rod are nonlinear. To see this, let us choose d as our unknown and construct the following residual $R(d)$ for our system

$$R(d) = \frac{A_0 E}{L_0}(d - \bar{d}) + F_c, \tag{1.50}$$

where F_c, the contact force between the obstacle and the rod (assumed positive in compression), is subject to the following constraints:

$$F_c \geq 0; \ g(d) \leq 0 \ \text{ and } \ F_c g(d) = 0. \tag{1.51}$$

Equations (1.51) are called *Kuhn-Tucker conditions* in mathematical parlance and physically require that the contact force be compressive, that the rod end not interpenetrate the obstacle, and that the contact force only be nonzero when $g = 0$; i.e., when contact between the rod and obstacle occurs. In fact, F_c is a Lagrange multiplier in this problem, enforcing the kinematic constraint (1.49). We see that the condition operating on the right end of the bar is neither a Dirichlet nor a Neumann boundary condition; in fact, both the stress and the displacement at this point are unknown but are related to each other through constraints (1.51).

Plots of the residual defined by (1.50) and (1.51) are given in Figure 1.5 for the two distinct cases of interest: where contact does not occur (i.e., when $\bar{d} < g_0$) and where contact does occur (when $\bar{d} \geq g_0$). The solutions (i.e., the zeros of R) are readily apparent. When no contact occurs,

$d = \bar{d}$, while in the case of contact, $d = g_0$. The internal stresses generated in the bar are then readily deduced.

One may note from Figure 1.5 some important practical features of this problem. First, in both cases the admissible region for d is restricted to be less than g_0. Second, at the value $d = g_0$, each diagram shows the residual to be multiple-valued, which is a direct consequence of the fact that in this condition (i.e., where $g = 0$), F_c can be any positive number.
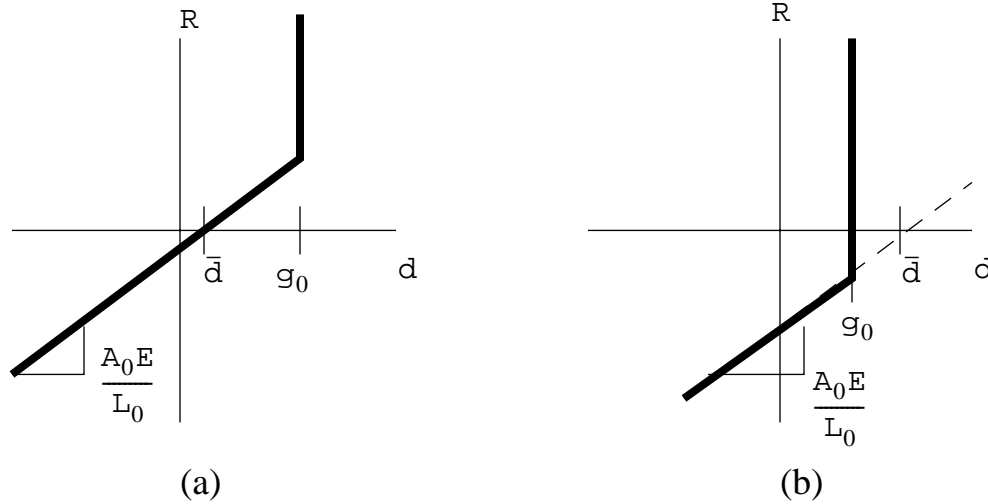
Figure 1.5   **Plots of residuals versus displacement for rigid obstacle problem: (a) the case where $\bar{d} < g_0$ (no contact) and (b) the case where $\bar{d} \geq g_0$ (contact).**

Finally, although the solution to our simple model problem is readily guessed, we can see from both cases that the plot of R versus d is only piecewise linear; the kink in each diagram indicates the fact that a finite tangent stiffness operates when contact is not active, changing to an infinite effective stiffness imposed by Eqs. (1.51) when contact between rod and obstacle is detected. This contact detection therefore becomes an important feature in general strategies for contact problems and introduces both nonlinearities and nonsmoothnesses into the global equations as this rather simple example demonstrates.

Linear Elastic IBVP

Introduction and Notation

Having reviewed some relevant nonlinearities in the context of an admittedly simple structural element, let us begin to generalize our problem description to encompass a larger group of continuous bodies. We begin this development by first reviewing the basic equations of linear elasticity, where we assume small motions and linear material behavior. This discussion will provide the basis for a more general notational framework in the next section where we will remove the kinematic restriction to small motions and also allow the material to behave in an inelastic manner.

The notation we will use in this section is summarized in Figure 1.6, where we have depicted a solid body positioned in three-dimensional Euclidean space or $\Re^3$. The set of spatial points $\mathbf{x}$ defining the body is denoted by Ω, and we consider the boundary $\partial\Omega$ to be subdivided into two regions, Γ_u and Γ_σ, where Dirichlet and Neumann boundary conditions will be specified as discussed below. We assume that these regions obey the following

$$\begin{aligned} \Gamma_u \bigcup \Gamma_\sigma &= \partial\Omega \\ \Gamma_u \bigcap \Gamma_\sigma &= \varnothing \end{aligned} \quad . \tag{1.52}$$

The unknown, or dependent variable, in this problem is $\mathbf{u}$, the vector-valued displacement which, in general, depends upon $\mathbf{x} \in \Omega$ and time t.
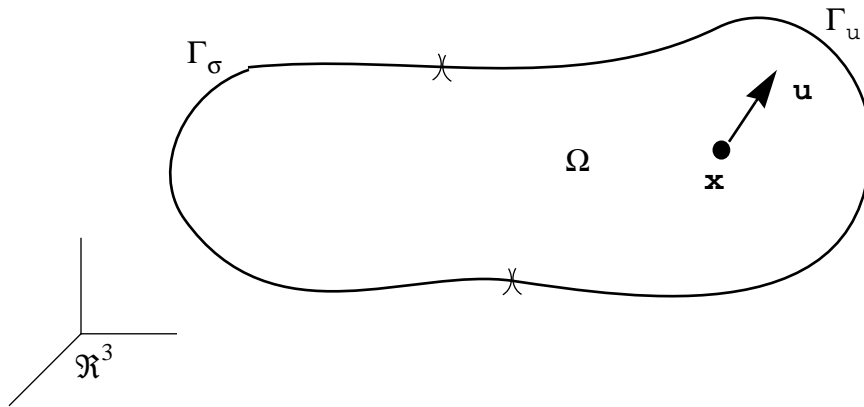
Figure 1.6   Notation for the linear elastic initial/boundary value problem.

Equations of Motion

At any point in Ω, the following statement of local linear momentum balance must hold:

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{1.53}$$

Note that $\nabla \cdot \mathbf{T}$ denotes the divergence operator applied to $\mathbf{T}$, the Cauchy stress tensor. The vector $\mathbf{f}$ denotes the distributed body force in Ω, with units of force per unit volume, and ρ denotes the mass density, which need not be uniform. Equation (1.53) represents the balance of linear momentum in so-called direct notation; balance of angular momentum is enforced by the assumption that the tensor $\mathbf{T}$ is symmetric. We will frequently employ *indicial notation* in the work that follows. Toward that end Eq. (1.53) can be reexpressed as

$$T_{ij,j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{1.54}$$

where indices i and j run between 1 and 3, and repeated indices within a term of an expression imply a summation over that index, that is:

$$T_{ij,j} = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j}. \tag{1.55}$$

One should take the notation $\beta_{,j}$ to indicate partial differentiation with respect to x_j. When using indicial notation repeated indices will always imply sums unless otherwise indicated.

As indicated above, the dependent variables are the u_i, so it is necessary to specify the relation between the displacements and the Cauchy stress. In linear elasticity this is accomplished by two additional equations. The first is the strain-displacement relation:

$$E_{ij} = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.56}$$

where the notation $u_{(i,j)}$ is used to denote the symmetric part of the displacement gradient. The second equation is the linear constitutive relation between T_{ij} and E_{ij}, which is normally written via

$$T_{ij} = C_{ijkl} E_{kl}. \tag{1.57}$$

Note that C_{ijkl} is the fourth-order elasticity tensor, to be discussed further below. Equations ,(1.56) and (1.57) can also be written in direct notation via

$$\mathbf{E} = \nabla_s \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \tag{1.58}$$

and

$$\mathbf{T} = \mathbf{C}{:}\mathbf{E}, \tag{1.59}$$

where the colon indicates double contraction of the fourth-order tensor $\mathbf{C}$ with the second-order tensor $\mathbf{E}$.

The fourth-order elasticity tensor $\mathbf{C}$ is ordinarily assumed to possess a number of symmetries that greatly reduce the number of independent components that describe it. It possesses *major symmetry*, which means $C_{ijkl} = C_{klij}$, and it is also assumed to have *minor symmetries*, meaning, for example, that $C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk}$. Another important property of the elasticity tensor is positive definiteness, which implies in this context that

$$A_{ij}C_{ijkl}A_{kl} > 0 \text{ for all symmetric tensors A} \tag{1.60}$$

$$A_{ij}C_{ijkl}A_{kl} = 0 \text{ iff A=0}. \tag{1.61}$$

In the most general case, assuming the aforementioned symmetries and no others, the elasticity tensor has 21 independent components. Various material symmetries reduce this number greatly, with the most specific case being given by an isotropic material that possesses rotational symmetry in all directions. In this case only two independent elastic constants are required to specify $\mathbf{C}$, which under these circumstances can be written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}], \tag{1.62}$$

where δ_{ij}, the *Kronecker delta*, satisfies

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}; \tag{1.63}$$

and λ and μ denote the *Lame parameters* for the material. These can be written in terms of the more familiar elastic modulus and Poisson's ratio via

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{1.64}$$

$$\mu = \frac{E}{2(1+\nu)}. \tag{1.65}$$

The quantity μ is also known as the *shear modulus* for the material.

Substitution of (1.58) and (1.59) into (1.53) gives a partial differential equation for the vector-valued unknown displacement field $\mathbf{u}$. Full specification of the problem at hand must include suitable boundary and initial conditions as discussed next.

Boundary and Initial Conditions

Paralleling the earlier discussion of the one-dimensional example, we will consider the possibility of two types of boundary conditions, Dirichlet and Neumann. Dirichlet boundary conditions will be imposed on the region Γ_u in Figure 1.6 as follows:

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) \forall \mathbf{x} \in \Gamma_u, t \in (0, T). \tag{1.66}$$

Note that $\bar{\mathbf{u}}(\mathbf{x}, t)$ denotes a prescribed displacement vector depending, in general, on spatial position and time. The simplest and perhaps most common example of such a boundary condition would be a fixed condition that, if imposed throughout the time interval of interest $(0, T)$ and for all of Γ_u, would imply $\bar{\mathbf{u}}(\mathbf{x}, t) = \mathbf{0}$.

The other type of boundary condition is a Neumann or traction boundary condition. To write such a condition, we must first define the concept of *traction* on a surface. If we use $\mathbf{n}$ to denote the outward normal to the surface Γ_u at a point $\mathbf{x} \in \Gamma_u$, the traction vector $\mathbf{t}$ at $\mathbf{x}$ is defined via

$$\mathbf{t} = \mathbf{Tn}, \tag{1.67}$$

or in indicial notation

$$t_i = T_{ij}n_j. \tag{1.68}$$

Physically this vector represents a force per unit area acting on the external surface at $\mathbf{x}$. A Neumann boundary condition is then written in the current notation as

$$\mathbf{T}((\mathbf{x}, t)\mathbf{n}(\mathbf{x})) = \bar{\mathbf{t}}(\mathbf{x}, t) \forall \mathbf{x} \in \Gamma_\sigma, t \in (0, T). \tag{1.69}$$

Note that $\bar{\mathbf{t}}(\mathbf{x}, t)$ is the prescribed traction vector field on $\Gamma_\sigma \times (0, T)$. One could identify several examples of such a boundary condition. An unfixed surface free of any external forcing would be described by $\bar{\mathbf{t}} = \mathbf{0}$. A surface subject to a uniform pressure loading p, on the other hand, could be described by setting $\bar{\mathbf{t}}(\mathbf{x}, t) = -p\mathbf{n}(\mathbf{x})$, where we assume a compressive pressure to be positive.

With these definitions in hand, we recall the restrictions (1.52) on Γ_u and Γ_σ and physically interpret them as follows: 1) one must specify either a traction or a displacement boundary condition at every point of $\partial\Omega$; and 2) at each point of $\partial\Omega$, one may not specify both the traction and the displacement but must specify one or the other. In fact, these conditions are slightly more stringent than required. For example, the problem remains well-posed if, for each component direction i, we specify either the traction component $\bar{t}_i$ or displacement component $\bar{u}_i$ at each point $\mathbf{x} \in \partial\Omega$ as long as for a given spatial direction, we do not attempt to specify both.

In other words, we may specify a displacement boundary condition in one direction at a point while specifying a traction boundary condition in the other. An example of such a case would be the common "roller" boundary condition where a point is free to move in a traction-free manner tangent to an interface (i.e., a traction boundary condition), while being constrained from movement in a direction normal to an interface (i.e., a displacement boundary condition). Of course, a multitude of other boundary condition permutations could be identified. Thus, while we choose a rather simple boundary condition restriction, summarized by (1.52), for notational simplicity, it is important to realize that many other possibilities exist and require only minor alterations of the methodology we will discuss.

The final important ingredient in our statement of the linear elastic problem is the specification of initial conditions. One may note that our partial differential equation (1.53) is second order in time; accordingly, two initial conditions are required. In the current context these amount to initial conditions on the displacement $\mathbf{u}$ and the velocity $\dot{\mathbf{u}}$ and can be rather straightforwardly specified via

$$\mathbf{u}(\mathbf{x}, 0) \ = \ \mathbf{u}_0(\mathbf{x}) \text{ on } \Omega \tag{1.70}$$

$$\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) \ = \ \mathbf{v}_0(\mathbf{x}) \text{ on } \Omega, \tag{1.71}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ are the prescribed initial displacement and velocity fields, respectively.

Problem Specification

We now collect the equations and conditions of the past two sections into a single problem statement for the linear elastic system shown in Figure 1.6. For the elastodynamic case this problem falls into the category of an *initial/boundary value problem*, since both types of conditions are included in its definition. Our problem is formally stated as follows:

Given the boundary conditions $\bar{\mathbf{t}}$ on $\Gamma_\sigma \times (0, T)$ and $\bar{\mathbf{u}}$ on $\Gamma_u \times (0, T)$, the initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ on Ω, and the distributed body force $\mathbf{f}$ on $\Omega \times (0, T)$, find the displacement field $\mathbf{u}$ on $\bar{\Omega} \times (0, T)$ such that:

$$\nabla \cdot \mathbf{T} + \mathbf{f} \ = \ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \text{ on } \Omega \times (0, T), \tag{1.72}$$

$$\mathbf{u}(\mathbf{x}, t) \ = \ \bar{\mathbf{u}}(\mathbf{x}, t) \text{ on } \Gamma_u \times (0, T), \tag{1.73}$$

$$\mathbf{t}(\mathbf{x}, t) \ = \ \bar{\mathbf{t}}(\mathbf{x}, t) \text{ on } \Gamma_\sigma \times (0, T), \tag{1.74}$$

$$\mathbf{u}(\mathbf{x}, 0) \,=\, \mathbf{u}_0(\mathbf{x}) \text{ on } \Omega\,, \tag{1.75}$$

$$\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) \,=\, \mathbf{v}_0(\mathbf{x}) \text{ on } \Omega\,, \tag{1.76}$$

where the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} \,=\, \mathbf{C}{:}(\nabla_s \mathbf{u})\,. \tag{1.77}$$

Equations (1.72) through (1.77) constitute a linear hyperbolic initial/boundary value problem for the dependent variable $\mathbf{u}$.

The Quasistatic Approximation

Before leaving the elastic problem, it is worthwhile to discuss how our problem specification will change if inertial effects are negligible in the equilibrium equations. This special case is often referred to as the *quasistatic assumption* and considerably simplifies specification of the problem.

Simply stated, the quasistatic assumption removes the second temporal derivative of $\mathbf{u}$ from (1.72) thereby eliminating the need for initial conditions (1.73) and (1.74). Such an approximation is appropriate when the loadings do not vary with time or when they vary over time scales very much longer than the periods associated with the fundamental structural modes of Ω. It is convenient, however, to maintain time in our description of the problem for two reasons: 1) the loadings $\bar{\mathbf{t}}$ and $\mathbf{f}$ and the displacement condition $\bar{\mathbf{u}}$ may still vary with time; and 2) when we consider more general classes of constitutive equations, we may wish to allow time dependence in the stress/strain response. Accordingly, we state below a boundary value problem appropriate for quasistatic response of a linear elastic system.

Given the boundary conditions $\bar{\mathbf{t}}$ on $\Gamma_\sigma \times (0, \text{T})$, $\bar{\mathbf{u}}$ on $\Gamma_u \times (0, \text{T})$, and the distributed body force $\mathbf{f}$ on $\Omega \times (0, \text{T})$, find the displacement field $\mathbf{u}$ on $\overline{\Omega} \times (0, \text{T})$ such that:

$$\nabla \cdot \mathbf{T} + \mathbf{f} \,=\, 0 \text{ on } \Omega \times (0, \text{T})\,, \tag{1.78}$$

$$\mathbf{u}(\mathbf{x}, t) \,=\, \bar{\mathbf{u}}(\mathbf{x}, t) \text{ on } \Gamma_u \times (0, \text{T})\,, \tag{1.79}$$

$$\mathbf{t}(\mathbf{x}, t) \,=\, \bar{\mathbf{t}}(\mathbf{x}, t) \text{ on } \Gamma_\sigma \times (0, \text{T})\,, \tag{1.80}$$

where the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} \,=\, \mathbf{C}{:}(\nabla_s \mathbf{u})\,. \tag{1.81}$$

We note in passing that given a time $t \in (0, T)$, Eqs. (1.78) through (1.81) constitute a linear, elliptic boundary value problem governing the dependent variable $\mathbf{u}$.

Weak Forms

Introduction

A key feature of the finite element method is the form of the boundary value problem (or initial/boundary value problem in the case of dynamics) that is discretized. More specifically the finite element method is one of a large number of variational methods that rely on the approximation of integral forms of the governing equations. In this section we briefly examine how such integral (alternatively, weak or variational) forms are constructed for the linear elastic system we have introduced.

Quasistatic Case

It proves convenient from notational and conceptual viewpoints to consider the quasistatic case first. Accordingly, we recall Eqs. (1.78) through (1.81) and explore an alternative manner in which these conditions can be stated. We consider a collection of vector-valued functions $\mathbf{w}$ that we call weighting functions for reasons that will soon be clear. We require that these functions $\mathbf{w}:\overline{\Omega} \to \Re^3$ satisfy

$$\mathbf{w} = \mathbf{0} \text{ on } \Gamma_u. \tag{1.82}$$

Furthermore, it is assumed that these functions are sufficiently smooth so that all partial derivatives can be computed. Suppose we have the solution $\mathbf{u}$ of Eqs. (1.78)-(1.81). We can then take any smooth function $\mathbf{w}$ satisfying (1.82) and compute its dot product with (1.78), which must produce

$$\mathbf{w} \cdot (\nabla \cdot \mathbf{T} + \mathbf{f}) = 0 \text{ on } \Omega \tag{1.83}$$

at each time $t \in (0, T)$. We can then integrate (1.83) over Ω to obtain

$$\int_{\Omega} \mathbf{w} \cdot (\nabla \cdot \mathbf{T} + \mathbf{f}) d\Omega = 0. \tag{1.84}$$

Equation (1.84) can be manipulated further by noting that

$$\mathbf{w} \cdot (\nabla \cdot \mathbf{T}) = \nabla \cdot (\mathbf{Tw}) - (\nabla \mathbf{w}):\mathbf{T} \tag{1.85}$$

(product rule of differentiation) and by also taking advantage of the divergence theorem from multivariate calculus:

$$\int_{\Omega} \nabla \cdot (\mathbf{Tw}) d\Omega = \int_{\partial\Omega} (\mathbf{n} \cdot \mathbf{Tw}) d\Gamma. \tag{1.86}$$

Note that $\mathbf{n}$ is the outward directed normal on $\partial\Omega$, and $\mathrm{d}\Gamma$ is a differential area of this surface. Use of Eqs. (1.85) and (1.86) in (1.84) and rearranging gives

$$\int_{\Omega}(\nabla\mathbf{w}){:}\mathbf{T}\mathrm{d}\Omega = \int_{\Omega}\mathbf{w}\cdot\mathbf{f}\mathrm{d}\Omega + \int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{Tw})\mathrm{d}\Gamma. \tag{1.87}$$

Taking advantage of the symmetry of $\mathbf{T}$ and noting, from Eq. (1.67), that the surface traction $\mathbf{t}$ equals $\mathbf{Tn}$, we can write:

$$\int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{Tw})\mathrm{d}\Gamma = \int_{\partial\Omega}(\mathbf{w}\cdot\mathbf{Tn})\mathrm{d}\Gamma = \int_{\partial\Omega}\mathbf{w}\cdot\mathbf{t}\mathrm{d}\Gamma. \tag{1.88}$$

We now recall restrictions (1.52), which tell us that $\partial\Omega$ is the union of Γ_{u} and Γ_{σ}. Since by definition $\mathbf{w} = \mathbf{0}$ on Γ_{u}, we can write

$$\int_{\partial\Omega}\mathbf{w}\cdot\mathbf{t}\mathrm{d}\Gamma = \int_{\Gamma_{\mathrm{u}}}\mathbf{w}\cdot\mathbf{t}\mathrm{d}\Gamma + \int_{\Gamma_{\sigma}}\mathbf{w}\cdot\mathbf{t}\mathrm{d}\Gamma = \int_{\Gamma_{\sigma}}\mathbf{w}\cdot\bar{\mathbf{t}}\mathrm{d}\Gamma, \tag{1.89}$$

where the last equality incorporates the boundary condition $\mathbf{t} = \bar{\mathbf{t}}$ on Γ_{σ}.

We collect these calculations to conclude

$$\int_{\Omega}(\nabla\mathbf{w}){:}\mathbf{T}\mathrm{d}\Omega = \int_{\Omega}\mathbf{w}\cdot\mathbf{f}\mathrm{d}\Omega + \int_{\Gamma_{\sigma}}\mathbf{w}\cdot\bar{\mathbf{t}}\mathrm{d}\Gamma, \tag{1.90}$$

which must hold for the solution $\mathbf{u}$ of Eqs. (1.78)-(1.81) for any $\mathbf{w}$ satisfying condition (1.82).

In order to complete our alternative statement of the boundary value problem, the concepts of solution and variational spaces need to be introduced. Let us define the *solution space* S_{t} corresponding to time t via

$$\mathrm{S}_{\mathrm{t}} = \{\mathbf{u}|\mathbf{u} = \bar{\mathbf{u}}(\mathrm{t})\text{ on }\Gamma_{\mathrm{u}}, \mathbf{u}\text{ is smooth}\} \tag{1.91}$$

and the *weighting space* W via

$$\mathrm{W} = \{\mathbf{w}|\mathbf{w} = \mathbf{0}\text{ on }\Gamma_{\mathrm{u}}, \mathbf{w}\text{ is smooth}\}. \tag{1.92}$$

With these two collections of functions in hand, let us consider the following alternative statement of the boundary value problem summarized by Eqs.(1.78)-(1.81):

Given the boundary conditions $\bar{\mathbf{t}}$ on $\Gamma_{\sigma}\times(0, \mathrm{T})$, $\bar{\mathbf{u}}$ on $\Gamma_{\mathrm{u}}\times(0, \mathrm{T})$, and the distributed body force $\mathbf{f}$ on $\Omega\times(0, \mathrm{T})$, find the $\mathbf{u}\in\mathrm{S}_{\mathrm{t}}$ for each time $\mathrm{t}\in(0, \mathrm{T})$ such that:

$$\int_{\Omega} (\nabla \mathbf{w}):\mathbf{T} d\Omega = \int_{\Omega} \mathbf{w} \cdot \mathbf{f} d\Omega + \int_{\Gamma_\sigma} \mathbf{w} \cdot \bar{\mathbf{t}} d\Gamma \qquad (1.93)$$

for all $\mathbf{w} \in \mathtt{W}$, where $\mathtt{S}_t$ is as defined in (1.91), $\mathtt{W}$ is as defined in (1.92), and where the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{C}:(\nabla_s \mathbf{u}). \qquad (1.94)$$

Since it explicitly requires only a weighted integral of the governing partial differential equation to be zero, rather than the differential equation itself, this statement of the boundary value problem is often referred to as a weak formulation.

Based upon the above derivation of the weak form, it should be clear that the solution $\mathbf{u}$ of Eqs. (1.78)-(1.81) (sometimes referred to as the *strong form*) will satisfy our alternative statement summarized by Eqs. (1.93) and (1.94). Less clear is the fact that solutions of the weak form will satisfy the strong form, as must be true for the two problem statements to be truly equivalent. Although not established here this equivalency can be rigorously established; the interested reader should consult [Hughes, T.J.R., 1987] for details. In the present discussion we simply remark that the equivalent argument depends crucially on the satisfaction of (1.93) for all $\mathbf{w} \in \mathtt{W}$, with the arbitrariness of $\mathbf{w}$ rendering the two statements completely equivalent.

Peeking ahead to numerical strategies, we can also remark that approximate methods will in effect narrow our definitions of the solution and weighting spaces to so-called *finite-dimensional* subspaces. Simply stated, this means that rather than including the infinite number of smooth $\mathbf{u}$ and $\mathbf{w}$ satisfying the requisite boundary conditions in our problem definition, we will restrict our attention to some finite number of functions comprising subsets of $\mathtt{S}_t$ and $\mathtt{W}$.

In so doing we introduce a difference between the solution of our (now approximate) weak form and the strong form, where the degree of approximation is directly related to the difference between the full solution and weighting spaces and the subsets of them used in the numerical procedure.

Finally, it is worthwhile at this point to make a connection to so-called *virtual work* methods that may be more familiar to readers versed in linear structural mechanics. In this derivation we will work in indicial notation so that the meaning of the direction notation statements above can be reinforced. Accordingly, for a possible solution $\mathtt{u}_i$ of the governing equations, let us write the following expression for the total potential energy of the system at hand:

$$\begin{aligned}
\mathtt{P}(\mathtt{u}_i) = & \frac{1}{2}\int_{\Omega} \mathtt{u}_{(i,\,j)} \mathtt{C}_{ijkl} \mathtt{u}_{(k,\,l)} d\Omega \\
& - \left[\int_{\Omega} \mathtt{u}_i \mathtt{f}_i d\Omega - \int_{\Gamma_\sigma} \mathtt{u}_i \bar{\mathtt{t}}_i d\Gamma \right].
\end{aligned} \qquad (1.95)$$

Note that the first term on the right-hand side represents the total *strain energy* associated with u_i, and the last two terms represent the potential energy of the applied loadings f_i and $\bar{t}_i$. A *virtual work principle* for this system simply states that the potential energy defined in (1.95) should be minimized by the equilibrium solution. Accordingly, let u_i now represent the actual equilibrium solution. We can represent any other, kinematically admissible displacement field via $u_i + \varepsilon w_i$, where ε is a scalar parameter (not necessarily small), and w_i is a so-called virtual displacement that we assume to obey the boundary conditions outlined in (1.82). This restriction on the w_i causes $u_i + \varepsilon w_i$ to satisfy the Dirichlet boundary conditions (hence the term "kinematically admissible") because the solution u_i does. We can write the total energy associated with any of these possible solutions via

$$
\begin{aligned}
P(u_i + \varepsilon w_i) = &\frac{1}{2}\int_{\Omega} (u_{(i,\,j)} + \varepsilon w_{(i,\,j)}) C_{ijkl}(u_{(k,\,l)} + \varepsilon w_{(k,\,l)}) d\Omega \\
&- \int_{\Omega}(u_i + \varepsilon w_i) f_i d\Omega - \int_{\Gamma_\sigma}(u_i + \varepsilon w_i)\bar{t}_i d\Gamma
\end{aligned}
\qquad (1.96)
$$

We now note that if the potential energy associated with u_i is to be lower than that of any other possible solution $u_i + \varepsilon w_i$, then the derivative of $P(u_i + \varepsilon w_i)$ with respect to ε at $\varepsilon = 0$ (i.e., at the solution u_i) should be zero for any w_i satisfying conditions (1.82), since u_i is an extremum point of the function P. Computing this derivative of (1.96) and setting the result equal to zero yields

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon = 0} P(u_i + \varepsilon w_i) = \left[\begin{array}{c} \displaystyle\int_{\Omega} w_{(i,\,j)} C_{ijkl} u_{(k,\,l)} d\Omega \\[2mm] \displaystyle -\int_{\Omega} w_i f_i d\Omega - \int_{\Gamma_\sigma} w_i \bar{t}_i d\Gamma \end{array} \right] = 0,
\qquad (1.97)
$$

which must hold for all w_i satisfying the boundary condition on Γ_u. Equation (1.97) can be manipulated further by noting that

$$
\begin{aligned}
w_{(i,\,j)} C_{ijkl} u_{(k,\,l)} &= w_{(i,\,j)} C_{ijkl} E_{kl} \\
&= w_{(i,\,j)} T_{ij} = w_{i,\,j} T_{ij}
\end{aligned}
\qquad (1.98)
$$

The last equality in (1.98), while perhaps not intuitively obvious, holds because of the symmetry of T_{ij}:

$$w_{(i,j)}T_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i})T_{ij}$$

$$= \frac{1}{2}(w_{i,j}T_{ij} + w_{j,i}T_{ji}).$$ (1.99)

$$= w_{i,j}T_{ij}$$

Use of (1.98) in (1.97) yields

$$\int_{\Omega}w_{i,j}T_{ij}d\Omega - \int_{\Omega}w_{i}f_{i}d\Omega - \int_{\Gamma_{\sigma}}w_{i}\bar{t}_{i}d\Gamma = 0,$$ (1.100)

which is seen to be nothing more than the indicial notation counterpart of (1.93). Summarizing, we see that the weak or integral form of the governing equations developed previously can be interpreted as a statement of the principle of minimum potential energy. It is because of this alternative viewpoint that the weighting functions w_i are sometimes called variations or virtual displacements, with the terminology used often depending upon the mathematical and physical arguments used to develop the weak form.

Despite the usefulness of this physical interpretation, it should be noted that the presence of an energy principle is somewhat specific to the case at hand and may be difficult or impossible to deduce for many of the nonlinear systems to be considered in our later study. For example, many systems are not conservative, including those featuring inelasticity, so at best our thermodynamic understanding must be expanded if we insist on formulating such problems in terms of energy principles. Thus while the energy interpretation is enlightening for many systems, including those featuring elastic continuum and/or structural response, insistence on this approach for more general applications of variational methods can be quite limiting. It is noteworthy, for example, that the derivation given in Eqs. (1.83)-(1.90) depended in no way upon the system being conservative or even upon the form of the constitutive equation used. We will exploit the generality of this weighted residual derivation as we increase the level of nonlinearity and complexity in the chapters to come.

Fully Dynamic Case

Another advantage of the weighted residual approach is that it can be straightforwardly applied to dynamic problems. Before examining the dynamic case in detail, whose development parallels that of quasistatic problems, it is worthwhile to emphasize again the definitions of the weighting and solution spaces and to highlight the differences between them. Examining the definition of S_t in (1.91) and that of W in (1.92), we see that S_t depends on t through the boundary conditions on Γ_u, while W is independent of time. We retain these definitions in the current cas, and pose the following problem corresponding to the elastodynamic system posed in the last section:

Given the boundary conditions $\bar{\mathbf{t}}$ on $\Gamma_\sigma \times (0, \mathtt{T})$ and $\bar{\mathbf{u}}$ on $\Gamma_u \times (0, \mathtt{T})$, the initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ on Ω, and the distributed body force $\mathbf{f}$ on $\Omega \times (0, \mathtt{T})$, find the $\mathbf{u} \in \mathtt{S}_t$ for each time $\mathtt{t} \in (0, \mathtt{T})$ such that:

$$\int_\Omega \rho \mathbf{w} \cdot \frac{\partial^2 \mathbf{u}}{\partial t^2} d\Omega + \int_\Omega (\nabla \mathbf{w}){:}\mathbf{T} d\Omega = \int_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega + \int_{\Gamma_\sigma} \mathbf{w} \cdot \bar{\mathbf{t}} d\Gamma \qquad (1.101)$$

for all $\mathbf{w} \in \mathtt{W}$, where $\mathtt{S}$ is as defined in (1.91), $\mathtt{W}$ is as defined in (1.92), and where the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{C}{:}(\nabla_s \mathbf{u}). \qquad (1.102)$$

In addition, the solution $\mathbf{u}$ is subject to the following conditions at $\mathtt{t} = 0$:

$$\int_\Omega \mathbf{w} \cdot (\mathbf{u}(0) - \mathbf{u}_0) d\Omega = 0 \qquad (1.103)$$

and

$$\int_\Omega \mathbf{w} \cdot \left(\frac{\partial \mathbf{u}}{\partial t}(0) - \mathbf{v}_0\right) d\Omega = 0, \qquad (1.104)$$

both of which must hold for all $\mathbf{w} \in \mathtt{W}$.

The integral form of the dynamic equations given in (1.101) is obtained just as was done in the quasistatic case, by taking the dynamic governing partial differential equation (1.72) and multiplying it by a weighting function, integrating over the body, and applying integration by parts to the stress divergence term. The new ingredients in the current specification are the initial conditions summarized by Eqs. (1.103) and (1.104) but should be recognized by the reader as simple weighted residual expressions of the strong form of the initial conditions given in Eqs. (1.75) and (1.76).

Before leaving this section, we reemphasize the fact that the weighting functions are time independent while the solution spaces remain time dependent. This fact will have important consequences later when numerical algorithms are discussed because, in effect, we will wish to use the same classes of functions in our discrete representations of $\mathtt{W}$ and $\mathtt{S}_t$. These discretizations will involve spatial approximation, which in the case of $\mathtt{S}_t$, will leave the time variable continuous in the discrete unknowns of the system to be solved.

This *semidiscrete approach* to transient problems is pervasive in computational mechanics and has its origin in the current context in the fundamental difference between the weighting and solution spaces.
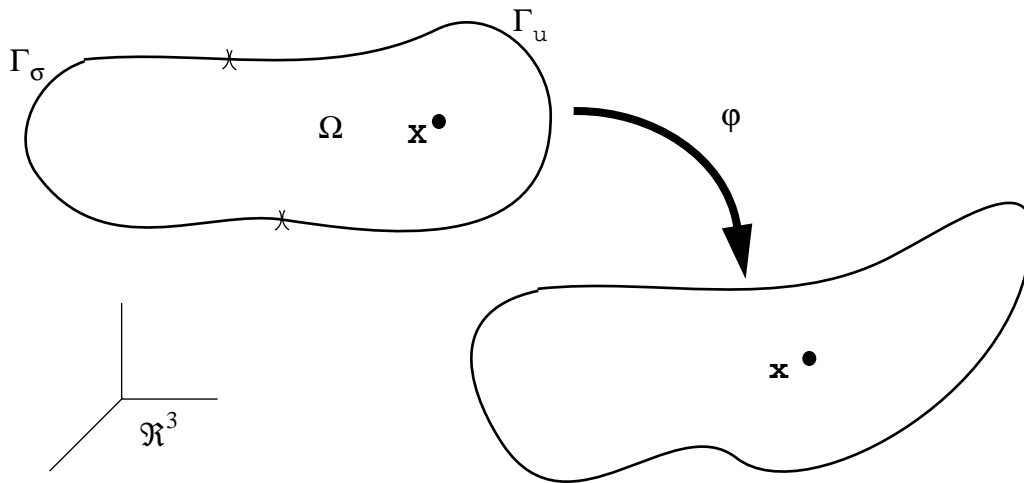
Large Deformation Problems

Introduction

In this section we extend our discussion of the linear elastic problem to accommodate two important features: potentially large motions and deformations, and nonlinear material response. We will do this by introducing a more general notational framework in which we will work throughout the text and then by examining in a fairly nonrigorous fashion how, provided certain concepts are kept in mind, the equations governing large deformation initial/boundary value problems are similar in form to their familiar counterparts from the small deformation theory. Rigorous prescription and understanding of large deformation problems can only be achieved through a careful examination of the concepts of nonlinear continuum mechanics, which will be the concern of the next chapter.

Notational Framework

The basic system we wish to consider is depicted schematically in Figure 1.7. We consider a body, initially in a location denoted by Ω, undergoing a time-dependent motion φ that describes its trajectory through the ambient space (assumed here to be $\Re^3$). The set Ω is called the *reference configuration* and can be thought of as consisting of points $\mathbf{x}$ that serve as labels for the material points existing at their respective locations.

Figure 1.7   Notation for large deformation initial/boundary value problems.

For this reason the coordinates $\mathbf{x}$ are often called *reference* or *material coordinates*. We assume, as before, that the surface $\partial\Omega$ of Ω can be decomposed into subsets Γ_σ and Γ_u, obeying restrictions (1.52). The general interpretation of these surfaces remains the same: traction

boundary conditions will be imposed on Γ_σ, and displacement boundary conditions will be imposed on Γ_u. Full specification of these boundary conditions must be deferred, however, until some continuum mechanical preliminaries are discussed.

We have mentioned that the motion φ is, in general, time dependent. In fact, we could write this fact in mathematical terms as $\varphi : \overline{\Omega} \times (0, \mathtt{T}) \rightarrow \Re^3$. If we fix the time argument of φ, we obtain a *configuration mapping* φ_t, summarized as $\varphi_t : \overline{\Omega} \rightarrow \Re^3$, which gives us the location of the body at time $\mathtt{t}$ given the reference configuration Ω. Coordinates in the current location $\varphi_t(\Omega)$ of the body will be denoted by $\mathbf{x}$.

The current location is often called the *spatial configuration* and the coordinates, $\mathbf{x}$, *spatial coordinates*. Given a material point $\mathbf{X} \in \Omega$ and a configuration mapping φ_t, we may write

$$\mathbf{x} = \varphi_t(\mathbf{X}).\tag{1.105}$$

A key decision in writing the equations of motion for this system is whether to express the equations in terms of $\mathbf{X} \in \Omega$ or $\mathbf{x} \in \varphi_t(\Omega)$.

Lagrangian and Eulerian Descriptions

The choice of whether to use reference coordinates $\mathbf{X}$ or spatial coordinates $\mathbf{x}$ in the problem description is generally highly dependent on the physical system to be studied. For example, suppose we wish to write the equations of motion for a gas flowing through a duct or for a fluid flowing through a nozzle. In these cases the physical region of interest (the control volume bounded by the duct or nozzle) is fixed and does not depend on the solution or time.

It could also be observed that identification of individual particle trajectories in such problems is probably not of primary interest, with such quantities as pressure, velocity, temperature, and so forth at particular locations in the flow field being more desirable. In such problems it is generally most appropriate to associate field variables and equations with spatial points or in the current notation, points $\mathbf{x}$. A system described in this manner is said to be utilizing the *Eulerian description* and implicitly associates all field variables and equations with spatial points $\mathbf{x}$ without specific regard for the material points $\mathbf{X}$ involved in the flow of the problem. Most fluid and gas dynamics problems are written this way, as are problems in hydrodynamics and some problems in solid mechanics involving fully developed plastic flow.

When thinking of Eulerian coordinate systems, it is sometimes useful to invoke the analogy of watching an event through a window: the window represents the Eulerian frame and has our coordinate system attached to it. Particles pass through our field of view thereby defining a flow,

but we describe this flow from the frame of reference of our window without specific reference to the particles undergoing the motion we observe.

In most solid mechanics applications, by contrast, the identity of specific material particles is of central interest in modeling a system. For example, the plastic response of metals is history dependent, meaning that the current relationship between stress and strain at a point in the medium depends on the deformation history associated with that material point. To use such models effectively requires knowledge of the history of individual particles, or material points, throughout a deformation process. Furthermore, many physical processes we wish to describe do not lend themselves to an invariant Eulerian frame: in a forging process, for example, the metal at the end of the procedure occupies a very different region in space than it did at the outset. For these and other reasons, the predominant approach to solid mechanics systems is to write all equations in terms of material coordinates or to use the *Lagrangian* frame of reference.

Returning to the notation summarized in Figure 1.7, we associate all field variables and equations with points $\mathbf{X} \in \Omega$ and keep track of these reference particles throughout the process. One may note, in the last subsection, a bias toward this approach already; namely, we have written the primary unknown in the problem (φ) as a function of $\mathbf{X} \in \Omega$ and $t \in (0, T)$.

Governing Equations in the Spatial Frame

With the above discussion as background, we turn now to the equations of motion governing the motion of a medium. Interestingly if we adopt for the moment the spatial frame as our frame of reference, the form of these equations is largely unchanged from the linear elastic case presented previously. Let us fix our attention on some time $t \in (0, T)$ and consider the current location (unfortunately unknown to us) of the body Ω. Over this region $\varphi_t(\Omega)$, the following conditions must hold:

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \rho \mathbf{a} \text{ on } \varphi_t(\Omega), \tag{1.106}$$

$$\varphi_t = \overline{\varphi_t} \text{ on } \varphi_t(\Gamma_u), \tag{1.107}$$

and

$$\mathbf{t} = \bar{\mathbf{t}} \text{ on } \varphi_t(\Gamma_\sigma), \tag{1.108}$$

subject, of course, also to initial conditions at $t = 0$. Some explanation of these equations is necessary. The nabla operator ∇ in (1.106) is to be interpreted as being with respect to spatial coordinates $\mathbf{x}$.

The acceleration $\mathbf{a}$ is referred to spatial coordinates but is the (material) acceleration of the particle currently at $\mathbf{x}$, and $\overline{\varphi_t}$ is to be interpreted as a given or prescribed location for particles

on the Dirichlet boundary. We leave the constitutive law governing $\mathbf{T}$ unspecified at this point but remark that, in general, the stress must depend on φ_t through appropriate strain/displacement and stress/strain relations. What we see from Eqs. (1.106) through (1.108) is that the equations of motion are easily written in the form inherited from the kinematically linear case but that the frame in which this is done, the spatial frame, is not independent of the unknown field φ_t but relies upon it for its own definition.

Thus although the equations we now consider are essentially identical in form to those from linear elasticity, they possess a considerably more complex relationship to the dependent variable. As will be provided in the next chapter, full and rigorous specification of this more general boundary value problem requires an in-depth treatment of the continuum mechanics of large deformations.

Before leaving this topic, an item that frequently causes confusion should be addressed. Although we have written the governing equations in (1.106) through (1.108) in terms of the spatial domain, this does not imply an Eulerian statement of the problem at hand. In fact, if we choose (as we have done) to consider our dependent variable (in this case φ_t) to be a function of reference coordinates, the framework we choose is inherently Lagrangian. Another way of saying this is that Eqs. (1.106)-(1.108) are the Lagrangian equations of motion which have been converted through a change-of-variables so that they are written in terms of $\mathbf{x}$. In the remainder of this text, the reader should assume a Lagrangian framework unless otherwise noted.

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